Math 358 – Foundations of Mathematics **Fall 2008**
11-13-08 Class Notes

Section 5.1
5.1: 1b, 1c, 4, 6, 8, 9

Functions

Recall that a relation on a set S is any subset of $S \times S$. More generally, a relation is a subset of $X \times Y$ where X and Y are any sets (possibly the same set).

Definition: A ***function*** is a relation in which no distinct elements have the same first coordinate. If f is a function, the domain of f is denoted by $Dom(f)$. If a function f (which is a subset of $X \times Y$) has $Dom(f) = X$, we often write $f : X \to Y$ and say that f maps X ***into*** Y. Notice this notation therefore implies that the domain is X and the range is contained in Y (but is not necessarily equal to Y).

Example 1:
(a) Let $X = \{1,2,3,4\}$ and $Y = \{2,4,6,8,10\}$. Then
$f = \{(1,2),(2,4),(3,6),(4,8)\}$ is a function mapping X into Y.
(b) Let $X = \{1,2,3,4,5\}$ and $Y = \{2,4,6\}$. Then
$f = \{(1,2),(2,2),(3,6),(4,6),(5,4)\}$ is a function mapping X into Y.
(c) Let $X = \mathbb{N}$ and $Y = \mathbb{R}$. Then $f = \{(n,x) : x = \sqrt{n}\}$ is a function mapping X into Y.

In a function, $[x]$ is always a singleton, but we rarely write $[x] = \{y\}$. We usually denote this by the more common notation $y = f(x)$. Also, we often define functions using equations: "Define $f : \mathbb{N} \to \mathbb{R}$ by $f(n) = \sqrt{n}$" describes the function in Example 1(c).

Now, let $f : X \to Y$ be a function. As noted in the definition, no two distinct elements of f can have the same first coordinate. Also, the range of f is a subset of Y. Functions which have no two distinct elements with the same *second* coordinate have a special name, as do functions whose range *is* Y.

Definition: Let $f : X \to Y$ be a function. If the range of f is Y, then f is said to map X ***onto*** Y. Also, we say that f is ***surjective*** (or a ***surjection***). If no two distinct elements have the same second coordinate, f is called a ***one-***

to-one function (or ***injective*** or an ***injection***). If a function is injective and surjective, we say it is ***bijective*** (or a ***bijection***).

Another way to characterize injective functions is:

A function f is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Example 2: The function in Example 1(a) is injective but not surjective.
The function in Example 1(b) is surjective but not injective.
The function in Example 1(c) is a bijection.

Example 3: Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x + 1$. Is f injective, surjective, bijective, or none of the above?

To determine injectivity, we'll use the condition in the box. Suppose $f(x_1) = f(x_2)$. That implies that

$$
\begin{aligned}
2x_1 + 1 &= 2x_2 + 1 \\
2x_1 &= 2x_2 \\
x_1 &= x_2
\end{aligned}
$$

So f is injective.

Determining surjectivity is a little more involved. To show that f is surjective, you need choose an arbitrary element of $\mathbb{Z}$, and find the element that gets mapped to it. To show that f is not surjective, you need to find an element of $\mathbb{Z}$ that nothing gets mapped to. Of course, only one of these procedures is possible on any given problem.

Since I think f in this case is *not* surjective, I need to find an element of $\mathbb{Z}$ that nothing gets mapped to. I notice that the image of any x is an odd integer. So I choose 2 and show that nothing gets mapped to 2. Suppose $f(x) = 2$ for some $x \in \mathbb{Z}$. Then

$$\begin{aligned} 2x + 1 &= 2 \\ 2x &= 1 \\ x &= \tfrac{1}{2} \end{aligned}$$

which is a contradiction. Therefore, f is injective, but not surjective.